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On the b-antighost in the Pure Spinor Quantization of Superstrings

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Abstract

Recently Berkovits has constructed a picture raised, compound field b_B which is used to compute higher loop amplitudes in the pure spinor approach of superstrings. On the other hand, in the twisted and gauge fixed, superembedding approach with $n = 2$ world-sheet (w.s.) supersymmetry that reproduces the pure spinor formulation, a field b appears quite naturally as the current of one of the two twisted charges of the w.s. supersymmetry, the other being the BRST charge. In this paper we study the relation between b and b_B. We shall show that bZ, where Z is a picture raising operator, and b_B belong to the same BRST cohomological class. This result is of importance since it implies that the cumbersome singularity which is present in b, is in fact harmless if b is combined with Z.

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The pure spinor approach, developed by Berkovits in [1]-[5], provides a consistent quantization scheme for superstring theories with manifest, super-Poincare covariance. Whereas untill recently only the prescription to compute tree level amplitudes was known, now in an important paper [6] the general prescription for calculating higher genus amplitudes has also been proposed. Then we could say that the pure spinor approach provides a consistent alternative to the well-known NSR and GS formulations which shares the advantages of both formulations without their disadvantages.

To compute higher loop amplitudes in superstring theories, a key ingredient is provided by insertions of a field b, with ghost number -1 which satisfies the equation

$$\Big\{Q, b\Big\} = T, \tag{1}$$

where Q and T are the BRST charge and the stress-energy tensor, respectively. In the pure spinor approach Q and T are given by

$$\begin{aligned} Q &= \frac{1}{2\pi i}\int dz \lambda^\alpha d_\alpha = \oint (\lambda d), \\ T &= -\frac{1}{2}\Pi^a\Pi_a - d_\alpha \partial\theta^\alpha + \omega_\alpha \partial\lambda^\alpha. \end{aligned} \tag{2}$$

Here Π^a and d_α are respectively the covariant momenta of the superspace coordinates X^a and θ^α, the ghost λ^α is a pure spinor, that is, a commuting spinor with the constraint $\lambda\Gamma^a\lambda = 0$ and ω_α is its conjugate momentum. The action is the free field action of X, θ, λ and their momenta and, as a consequence of the pure spinor condition, it is invariant under the local symmetry

$$\omega' = \omega + \Gamma^a q_a \lambda, \tag{3}$$

where q^a are local gauge parameters. In the NSR (or GS) formulation b is the antighost of diffeomorphisms. On the other hand, in the pure spinor quantization the diffeomorphism ghosts are absent and to find a suitable b is a non-trivial task.

In an attempt [7] to understand the geometrical origin of the pure spinor approach it has been shown (classically and for the heterotic string) that the pure spinor formalism can be recovered as a twisted and gauge fixed version of the superembedding formulation of the string with $n = 2$ world-sheet (w.s.) supersymmetry. In this framework the existence of the pure spinor λ and the absence of diffeomorphism ghosts can be understood quite naturally. Moreover the BRST charge of Berkovits (see also [8]) is just one of the two twisted charges of the original $n = 2$ supersymmetry and the field b can be identified with the twisted current of the other charge. In this formulation, the b-ghost is of form

$$b = \frac{1}{2}(Y\Gamma_a\Pi^a d) + (\tilde{\omega}\partial\theta), \tag{4}$$

which indeed satisfies eq. (1). Here we have defined Y_α as $Y_\alpha = \frac{v_\alpha}{(v\lambda)}$ so that

$$(Y\lambda) = 1, \tag{5}$$

with v_α being constant [9]. Moreover we have also defined

$$\tilde{\omega}_\alpha = (\delta_\alpha^\beta - K_\alpha{}^\beta)\omega_\beta, \tag{6}$$

where the projector K takes the form

$$K_\alpha{}^\beta = \frac{1}{2}(\Gamma_a\lambda)_\alpha(Y\Gamma^a)^\beta. \tag{7}$$

In this article, we adopt the following conventions: the BRST transformation is of form

$$\Big\{Q, \partial\theta^\alpha\Big\} = \partial\lambda^\alpha,\ \Big\{Q, d_\alpha\Big\} = -\Pi^m(\Gamma_m\lambda)_\alpha,\ \Big[Q, \Pi^m\Big] = \lambda\Gamma^m\partial\theta,\ \Big[Q, \tilde{\omega}_\alpha\Big] = -\tilde{d}_\alpha, \tag{8}$$

where $\tilde{d} = (1-K)d$ (as for ω). And the curly bracket denotes the anti-commutator while the square one denotes the commutator.

An expression equivalent to (4), in U(5) notations, has been given in [4]. The non-Lorentz covariance of b, in eq. (4), (due to v_α) is not a problem since the Lorentz variation of b is BRST-exact but the singular behaviour of b for the configurations where Y_α diverges i.e. where $(v\lambda) = 0$, would be problematic.

Here it is worthwhile to note that the strategy in [6] is different where a "picture raised" b_B field is constructed such that, instead of (1), it satisfies the condition [3]

$$\Big[Q, b_B\Big] = TZ, \tag{9}$$

with Z being the "picture raising" operator given by

$$Z = \frac{1}{2}B_{mn}(\lambda\Gamma^{mn}d)\delta(B_{rs}N^{rs}). \tag{10}$$

Here B_{mn} is a constant, antisymmetric tensor and the Lorentz current N^{rs} is defined as

$$N^{rs} = \frac{1}{2}(\omega\Gamma^{rs}\lambda). \tag{11}$$

Then, we can easily show

$$\Big\{Q, Z\Big\} = 0. \tag{12}$$

Notice that N^{rs} and the ghost number current

$$j = (\omega\lambda), \tag{13}$$

(together with $(\omega\partial\lambda)$) are the only objects involving ω which are invariant under the local symmetry (3). The starting point of the Berkovits' construction of b_B is to consider the field [4]

$$G^\alpha = \frac{1}{2}\Pi^m(\Gamma_m d)^\alpha - \frac{1}{4}N_{rs}(\Gamma^{rs}\partial\theta)^\alpha - \frac{1}{4}j\partial\theta^\alpha, \tag{14}$$

[3] In this letter, products of field are considered at the same point. How to treat the generic case where T and Z are not inserted at the same point, has been shown in [6].

[4] We ignore the normal-ordering contributions thoughout this paper.

which satisfies

$$\Big\{Q, G^\alpha\Big\} = \lambda^\alpha T. \tag{15}$$

In this letter we shall show that eq. (4) with eq. (1) is equivalent to eq. (14) with eq. (15) and furthermore that from eq. (4) one can recover the Berkovits' construction of the "picture raised" b-field b_B. In particular we shall show that bZ belongs to the same BRST cohomological class as b_B. This result leads us to two important conclusions. One is to give support to the idea, advocated in [7], that a superembedding formulation with $n = 2$ w.s. supersymmetry could be at the origin of the pure spinor approach. The other conclusion is that insertions of the simpler compound field b given by (4) can be used in order to compute higher genus amplitudes if b is combined with the picture changing operator Z of eq. (10) (and a trivial cocycle is added) since then the singular behaviour of b and all the Y-dependence disappear from the amplitudes. It would be of some interest to compare the b-field of [6] and [7] with the b-field in the extended pure spinor formalism where the pure spinor condition is removed [10]-[14].

To verify the equivalence between eq. (4) and eq. (14), let us consider the identity

$$\lambda^\alpha\omega_\beta = \frac{1}{16}\delta^\alpha_\beta j - \frac{1}{16}(\Gamma^{rs})^\alpha{}_\beta N_{rs} + \frac{1}{384}(\Gamma^{rspq})^\alpha{}_\beta(\omega\Gamma_{rspq}\lambda). \tag{16}$$

If $\tilde{\omega}$ is rewritten as $\tilde{\omega}_\alpha = Y_\beta\lambda^\beta\omega_\alpha - \frac{1}{2}(Y\Gamma^a\omega)(\lambda\Gamma_a)_\alpha$, then one obtains from eq. (16)

$$\tilde{\omega}_\alpha = Y_\beta\Big(-\frac{1}{4}j\delta^\beta_\alpha - \frac{1}{4}N^{rs}(\Gamma_{rs})^\beta{}_\alpha\Big), \tag{17}$$

This result is not surprising since $\tilde{\omega}$ is invariant under the local symmetry eq. (3). With eq. (17), eq. (4) becomes

$$b = Y_\alpha G^\alpha, \tag{18}$$

and from eq. (1) one has

$$Y_\alpha\Big\{Q, G^\alpha\Big\} = Y_\alpha\lambda^\alpha T, \tag{19}$$

which coincide with eqs. (14) and (15) due to the arbitrariness of v_α.

Let us recall an important consequence of eq. (15), which was proved in [6]. For that it is convenient to introduce the following definitions: a tensor field $X_{\alpha_1\cdots\alpha_n}$ with n spinor indices will be called Γ_5-traceless if it vanishes when saturated with $(\Gamma_{a_1\cdots a_5})^{\alpha_i\alpha_{i+1}}$ between two adjacent indices. Moreover, a tensor field $Y^{\alpha_1\cdots\alpha_n}$ will be called pure Γ_5-trace if $Y^{\alpha_1\cdots\alpha_n} = \sum_{i=1}^{n-1} h_{(i)}^{\alpha_1\cdots((\alpha_i\alpha_{i+1}))\cdots\alpha_n}$ where $h_{(i)}^{\alpha_1\cdots((\alpha_i\alpha_{i+1}))\cdots\alpha_n}$ is symmetric in the indices α_i, α_{i+1} and $\Gamma^a_{\alpha_i\alpha_{i+1}} h_{(i)}^{\alpha_1\cdots((\alpha_i\alpha_{i+1}))\cdots\alpha_n} = 0$. Then in [6] it is shown that eq. (15) implies the existence of the fields $H^{\alpha\beta}$, $K^{\alpha\beta\gamma}$, $L^{\alpha\beta\gamma\delta}$ and $S^{\beta\gamma\delta}$, defined modulo pure Γ_5-trace terms, in such a way that

$$\Big[Q, H^{\alpha\beta}\Big] = (\lambda^\alpha G^\beta) + \cdots, \tag{20}$$

$$\Big\{Q, K^{\alpha\beta\gamma}\Big\} = (\lambda^\alpha H^{\beta\gamma}) + \cdots, \tag{21}$$

$$\Big[Q, L^{\alpha\beta\gamma\delta}\Big] = (\lambda^\alpha K^{\beta\gamma\delta}) + \cdots, \tag{22}$$

Moreover, since

$$\lambda^\eta L^{\alpha\beta\gamma\delta} = 0 + \cdots, \tag{23}$$

one obtains

$$L^{\alpha\beta\gamma\delta} = \lambda^\alpha S^{\beta\gamma\delta} + \cdots, \tag{24}$$

where the dots in equations (20)-(24) denote pure Γ_5-trace terms.

It is also convenient to notice that from eq. (10) one has

$$Z = \lambda^\beta Z_\beta, \tag{25}$$

and

$$\Big\{Q, Z_\beta\Big\} = \lambda^\gamma Z_{\gamma\beta}, \tag{26}$$

$$\Big[Q, Z_{\beta\gamma}\Big] = \lambda^\alpha Z_{\alpha\beta\gamma}, \tag{27}$$

and

$$\Big\{Q, Z_{\alpha\beta\gamma}\Big\} = \lambda^\eta Z_{\eta\alpha\beta\gamma} + \partial\lambda^\eta \Upsilon_{\eta\alpha\beta\gamma}. \tag{28}$$

Notice that all the Z's with more than one index and Υ are Γ_5-traceless.Altough this fact can be verified easily from (10), it also follows directly from (12) without knowing the explicit form of Z. This property is important since, as we shall see, the spinor indices of the fields $H^{\alpha\beta}$, $K^{\alpha\beta\gamma}$, $L^{\alpha\beta\gamma\delta}$ and $S^{\alpha\beta\gamma}$ are saturated with these of the Z's and Υ, and consequently the terms of Γ_5-trace class, which are left unspecified in these fields, do not contribute and are irrelevant.

After these preliminaries, we now turn our attention to eq. (1). With help of eqs. (12), (18) and (25), eq. (1) can be rewritten as

$$\begin{aligned} TZ &= \Big[Q, bZ\Big] \\ &= \Big[Q, \Big(Y_\alpha(G^\alpha\lambda^\beta - \lambda^\alpha G^\beta) + Y_\alpha\lambda^\alpha G^\beta\Big)Z_\beta\Big]. \end{aligned} \tag{29}$$

Since $G^\alpha\lambda^\beta - G^\beta\lambda^\alpha$ is obviously Γ_5-traceless one can use (20) so that, taking into account (26), eq. (29) reduces to

$$TZ = \Big[Q, Y_\alpha(H^{\alpha\beta} - H^{\beta\alpha})\lambda^\gamma Z_{\gamma\beta}\Big] + \Big[Q, b_1\Big], \tag{30}$$

where b_1 is defined as

$$b_1 = G^\beta Z_\beta. \tag{31}$$

Since we can see that the Γ_5-trace with respect to the indices γ, β in $(H^{\alpha\beta} - H^{\beta\alpha})\lambda^\gamma - H^{\gamma\beta}\lambda^\alpha$ does not contribute since $Z_{\gamma\beta}$ is Γ_5-traceless, we have using eqs. (21) and (27)

$$\begin{aligned} TZ &= \Big[Q, Y_\alpha\Big((H^{\alpha\beta} - H^{\beta\alpha})\lambda^\gamma - \lambda^\alpha H^{\gamma\beta}\Big)Z_{\gamma\beta}\Big] + \Big[Q, b_1 + b_2\Big], \\ &= \Big[Q, Y_\alpha(-K^{\alpha\beta\gamma} - K^{\gamma\beta\alpha} + K^{\gamma\alpha\beta})\lambda^\eta Z_{\eta\gamma\beta}\Big] + \Big[Q, b_1 + b_2\Big], \end{aligned} \tag{32}$$

where we have defined

$$b_2 = H^{\gamma\beta} Z_{\gamma\beta}. \tag{33}$$

Using (22) and (28), the same procedure can be repeated once again to get

$$TZ = \Lambda^{(a)} + \Lambda^{(b)} + \Big[Q, b_1 + b_2 + b_3\Big], \tag{34}$$

where we have defined

$$b_3 = -K^{\eta\gamma\beta} Z_{\eta\gamma\beta}, \tag{35}$$

and

$$\Lambda^{(a)} = \Big[Q, Y_\alpha(L^{\alpha\eta\gamma\beta} + L^{\eta\gamma\alpha\beta} - L^{\eta\gamma\beta\alpha} - L^{\eta\alpha\gamma\beta})\lambda^\epsilon Z_{\epsilon\eta\gamma\beta}\Big], \tag{36}$$

$$\Lambda^{(b)} = \Big[Q, Y_\alpha(L^{\alpha\eta\gamma\beta} + L^{\eta\gamma\alpha\beta} - L^{\eta\gamma\beta\alpha} - L^{\eta\alpha\gamma\beta})\partial\lambda^\epsilon \Upsilon_{\epsilon\eta\gamma\beta}\Big]. \tag{37}$$

$\Lambda^{(a)}$ can furthermore be rewritten as

$$\Lambda^{(a)} = \Big[Q, Y_\alpha\Big((L^{\alpha\eta\gamma\beta} + L^{\eta\gamma\alpha\beta} - L^{\eta\gamma\beta\alpha} - L^{\eta\alpha\gamma\beta})\lambda^\epsilon - L^{\epsilon\eta\gamma\beta}\lambda^\alpha\Big)Z_{\epsilon\eta\gamma\beta}\Big] + \Big[Q, b_4^{(a)}\Big], \tag{38}$$

where we have defined

$$b_4^{(a)} = L^{\epsilon\eta\gamma\beta} Z_{\epsilon\eta\gamma\beta}. \tag{39}$$

Since $Z_{\epsilon\eta\gamma\beta}$ is Γ_5-traceless, the first term in (38) vanishes owing to (23). Accordingly, $\Lambda^{(a)}$ is expressed in terms of a BRST-exact term

$$\Lambda^{(a)} = \Big[Q, b_4^{(a)}\Big]. \tag{40}$$

As for $\Lambda^{(b)}$ it can be written as

$$\Lambda^{(b)} = \Big[Q, b_4^{(b)}\Big], \tag{41}$$

where we have defined

$$
\begin{aligned}
b_4^{(b)} &= Y_\alpha\Big[Y_\kappa\lambda^\kappa(L^{\alpha\eta\gamma\beta}+L^{\eta\gamma\alpha\beta}-L^{\eta\gamma\beta\alpha}-L^{\eta\alpha\gamma\beta})-Y_\kappa L^{\kappa\eta\gamma\beta}\lambda^\alpha\Big]\partial\lambda^\epsilon\Upsilon_{\epsilon\eta\gamma\beta} \\
&+ Y_\alpha Y_\kappa\lambda^\alpha L^{\kappa\eta\gamma\beta}\partial\lambda^\epsilon\Upsilon_{\epsilon\eta\gamma\beta}.
\end{aligned}
\tag{42}
$$

As before the first term vanishes and then using eqs. (5) and (24), one has

$$
b_4^{(b)} = S^{\eta\gamma\beta}\partial\lambda^\epsilon\Upsilon_{\epsilon\eta\gamma\beta}. \tag{43}
$$

Consequently, from eqs. (34), (40) and (41), we have recovered eq. (9) where

$$
b_B = b_1 + b_2 + b_3 + b_4^{(a)} + b_4^{(b)}. \tag{44}
$$

Note that b_1, b_2, b_3, $b_4^{(a)}$, $b_4^{(b)}$ are respectively given in eqs. (31), (33), (35), (39), (43) which are in complete agreement with the result of ref. [6]. Also notice that our construction shows that bZ and b_B belong to the same BRST cohomological class, as promised. A related but alternative and interesting recipe to compute one-loop amplitudes has been recently proposed in [15]. It is of interest to remark that if in eq. (28) one replace Z with the unintegrated vertex operator $V = \lambda^\beta V_\beta$ that it is needed at one loop level and then perform the same manipulations that lead from eq. (28) to eq. (43), one ends with eq. (5.25) of [15].

Now a remark is in order. As can be seen in our above derivation, $S^{\alpha\beta\gamma}$ depends on Y_α through its dependence on $\tilde{\omega}$, so one might worry that this dependence could remain even in $b_4^{(b)}$ from eq. (43). In that case, $b_4^{(b)}$ could become singular at $(v\lambda) = 0$, thereby inducing troublesome divergences in the loop amplitudes. However, luckily enough, this is not the case. To show that, let us notice that $\left[Q, b_4^{(b)}\right]$ must be regular from eqs. (9) and (44) (noting that T and the other terms of b_B are regular) and therefore is independent of Y_α. Moreover, on general ground and modulo harmless contributions with only regular j , N^{rs}, and $(\omega\partial\lambda)$, it turns out that $b_4^{(b)}$ is expressed by a linear combination of the following terms:

$$
\begin{aligned}
b_4^{(b)} &= xB_{pq}(\tilde{\omega}\Gamma^{pq}\partial\lambda)j + yB_{pq}(\tilde{\omega}\Gamma^{pqrs}\lambda)(\partial\lambda\Gamma_{rs}\tilde{\omega}) + zB_{pq}(\tilde{\omega}\Gamma^{prst}\lambda)(\tilde{\omega}\Gamma^{q}{}_{rst}\partial\lambda) \\
&+ vB_{pr}(\tilde{\omega}\Gamma^{rs}\partial\lambda)N_s{}^p + wB_{pq}(\tilde{\omega}\Gamma^{pqrs}\partial\lambda)N_{rs},
\end{aligned}
\tag{45}
$$

with x, y, z, v, and w being constants. By a repeated use of the Fierz identity and taking into account the pure spinor condition $\lambda\Gamma^a\lambda = 0$, the last term of this equation reduces to a combination of the first two terms in this equation and of the fourth one modulo harmless contributions and the fourth term reduces to the first one modulo harmless contributions.. Then $b_4^{(b)}$ reduces to a linear combination of the first 3 terms of this equation and regular contributions independent of Y_α. The BRST variations of the these 3 terms depend on Y_α in an independent way. Indeed the Y-dependence of the variation of the first term is proportional to $(\lambda\Gamma^{apq}\partial\lambda)$, that of the second term is proportional to $(\lambda\Gamma^{apqrs}\lambda)$ and that of the third one to $(\lambda\Gamma^{pqrst}\partial\lambda)$. Therefore the coefficients of these terms must vanish separately and $b_4^{(b)}$ is regular.

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